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ON THE INFLUENCE OF NON-SYMMETRICAL MODES ON THE BUCKLING OF SHALLOW SPHERICAL SHELLS UNDER UNIFORM PRESSURE

By R. R. Parmerter and Y. C. Fung
CALIFORNIA INSTITUTE OF TECHNOLOGY

SUMMARY

In view of the wide discrepancy between theoretical and experimental results that still exists, the problem of buckling of shallow spherical shells under uniform external pressure is reexamined with the introduction of non-symmetrical deflection modes. The basic idea is that, for a symmetrical shell under gradually increasing symmetrical loading, the deformation will be symmetrical up to and after buckling, but it is feasible that a non-symmetrical mode participates in the snap-through process. Including this possibility in the governing equations may lower the theoretical buckling load. Some analytical results are presented and compared with experiment.

SYMBOLS

$\lambda^2 = \sqrt{12(1-\mu^2)}\,\dfrac{b^2}{at}$ = geometrical shell parameter

b = radius of shallow spherical segment

a = radius of sphere

t = shell thickness

μ = Poisson's ratio

p = pressure

$q = p(\frac{a}{t})^2 \dfrac{\sqrt{3(1-\mu^2)}}{2E}$ = non-dimensional pressure

s = inplane radial displacement (symmetrical)

z = inplane radial displacement (non-symmetrical)

y = inplane tangential displacement

$w(r,\theta) = u(r) + v(r)\cos n\theta$ = assumed displacement perpendicular to shell

$\bar{u}(r) = (\frac{\lambda}{b})^2 a\ u(r)$ = non-dimensional displacement

E = Young's Modulus

INTRODUCTION

The problem of oil-canning or snap-through buckling of shallow spherical shells is one of considerable practical and theoretical interest. The initial investigation (Ref. 5) was undertaken to see if the classical continuous deformation concept of buckling would work if the non-linear, rather than linear equations were used, or whether it is necessary to follow Tsien's suggestion (Ref. 2) and introduce an artificial "energy criterion" for jumping, or distinguish and evaluate the "upper" and "lower" buckling loads as advocated by Friedrichs (Ref. 3) with the concomitant necessity of assessing any real shell buckling load to be somewhere in between. Several theoretical investigations followed (Refs. 7-15), but the theoretical program has not yet been satisfactorily completed. Considerable differences between theoretical predictions and experimental measurements (Refs. 4, 5, 16, 17) still exist and remain unexplained.

Taking a cue from the similar, but much simpler, problem of shallow arches (Ref. 6), we propose to investigate the influence of non-symmetrical modes and non-symmetrical initial imperfections on the snap-through buckling of shallow spherical shells. The idea is that for a symmetrical shell under a symmetrical loading, the deflection will be symmetrical before buckling, and symmetrical after buckling; but during the buckling process, or in initiating the buckling process, a non-symmetrical mode may participate. In the case of arches, the participation of non-symmetrical modes lowers the critical buckling load considerably when the arch-rise parameter λ is sufficiently large. We would like to see if the same is true for spherical shells. Furthermore, in arches, any small initial imperfection of the non-symmetrical type has large effect toward lowering the critical buckling load; the same might be true for spherical shells.

Grigolyuk (Ref. 18) has presented some formulas for the non-symmetrical deformation of spherical shells, but no results with respect to buckling were given. Recently, Gjelsvik and Bodner (Ref. 19) investigated the same problem by a single term approximation. Although interesting features are demonstrated, the solutions are numerically unreliable. The work to be presented herein was initiated in 1960, and is not yet completely finished. However, in

demonstrating (see Fig. 1) that for relatively small values of λ good agreement is obtained with Weinitschke and Budiansky's theoretical results (Ref. 13) and with the experimental deformation pattern of Kaplan and Fung (Ref. 5), some confidence is gained as to the correctness of the approach: for this reason it is thought worthwhile to present this progress report. Only initially perfect spherical shells will be discussed herein.

THE MATHEMATICAL PROBLEM

We consider a shallow portion of a spherical shell, clamped along a circular boundary, (Fig. 2), and subjected to uniform pressure on its top. We assume that the shell is so shallow that von Kármán's large deflection equations for slightly curved plates are applicable. These equations are given in Ref. 18 for cylindrical coordinates. We assume that the shell is initially a perfect spherical cap, and that the deflection is symmetrical except during buckling. In other words, as the symmetrical load deflection path is traced, we look for bifurcation points involving non-symmetrical states. If such a point can be found, it represents a possible way of initiating a non-symmetrical transition mode which can carry the shell into the buckled state. The existence of these points can be discovered by considering small non-symmetrical disturbances about the symmetrical, large deflection solution. In other words, the equations can be linearized with respect to the non-symmetrical terms.

We assume that the deflection can be represented by a function of the form

$$w(r,\theta) = u(r) + v(r)\cos n\theta \tag{1}$$

where $v(r) << u(r)$. When this assumption is inserted into von Kármán's equations we find that the stress function $F(r,\theta)$ should be represented in the form

$$F(r,\theta) = G(r) + H(r)\cos n\theta \tag{2}$$

Neglecting second order terms in $v(r)$ and $H(r)$, von Kármán's equations reduce to:

$$D\nabla^4 u = p + \frac{1}{r}(G_{rr}u_r + G_r u_{rr}) + \frac{1}{a}\left(G_{rr} + \frac{1}{r}G_r\right) \tag{3a}$$

$$\frac{\nabla^4 G}{Et} = -\frac{1}{r}\left(u_r u_{rr} + \frac{1}{a}u_r\right) - \frac{1}{a}u_{rr} \tag{3b}$$

$$\frac{D\nabla^4(v\cos n\theta)}{\cos n\theta} = \frac{1}{r}(u_r H_{rr} + v_r G_{rr} + H_r u_{rr} + v_{rr} G_r + \frac{1}{a} H_r)$$

$$- \frac{n^2}{r^2}(G_{rr} v + H u_{rr} + \frac{1}{a} H) + \frac{1}{a} H_{rr} \tag{3c}$$

$$\frac{\nabla^4(H\cos n\theta)}{Et\cos n\theta} = - \frac{1}{r}(u_r v_{rr} + u_{rr} v_r + \frac{1}{a} v_r)$$

$$+ \frac{n^2}{r^2}(u_{rr} v + \frac{1}{a} v) - \frac{1}{a} v_{rr} \tag{3d}$$

where $\nabla^4 = \frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial r} + \frac{4}{r^4}\frac{\partial^2}{\partial\theta^2}$

$$- \frac{2}{r^3}\frac{\partial^3}{\partial r\partial\theta^2} + \frac{2}{r^2}\frac{\partial^4}{\partial r^2\partial\theta^2} + \frac{1}{r^4}\frac{\partial^4}{\partial\theta^4}$$

and u, G, H, v are functions of r only.

Equations 3a, 3b are the usual non-linear symmetric equations, while 3c, 3d are the equations of a linearized non-symmetrical perturbation on the symmetrical solution. If the boundary conditions are symmetrical, equations 3c, 3d have the solution $v = H = 0$, i.e., the solution is symmetrical. However, for certain values of the load intensity q, with its corresponding functions u(r), G(r), Eqs. 3c, 3d may admit non-trivial solutions. We seek the condition on u(r), G(r) for eigensolutions to exist. The non-symmetrical buckling pressure p can then be determined from eq. 3a.

By treating the problem in this manner we have uncoupled the symmetrical equations from the non-symmetrical terms. Thus the symmetrical problem can be solved independently. We begin by solving 3a and 3b. The existing satisfactory solutions to these equations are all numerical, and have been used only to predict the buckling pressure. For our purpose we need expressions for u(r) and G(r) which can be used in the solution of 3c and 3d. We use Galerkin's method to obtain these functions.

Previous attempts to solve 3a, b by Galerkin's method have not proven too successful. We have attempted to pick a two term expression for u(r) which is capable of reproducing the experimental

mode shapes reported by Kaplan and Fung. For values of $\lambda < 8$ a satisfactory representation can be given by

$$u = u_1 \left(1 + \cos \frac{\pi}{b} r\right) + u_2 \left(1 - \cos \frac{2\pi}{b} r\right) \tag{4}$$

The assumed form of u satisfies the clamped boundary conditions $\frac{\partial u}{\partial r} = u = 0$ at $r = b$, and the symmetry condition $\frac{\partial u}{\partial r} = 0$ at $r = 0$.

Using Eq. 4 we can solve Eq. 3b for G. Three constants of integration arise which are determined by the conditions of finite stress at $r = 0$ and zero radial displacement in the plane of the edge of the cap, at $r = b$. We now substitute G and u into Eq. 3a, which will not in general be exactly satisfied, but can be satisfied approximately in the Galerkin sense. This leads to two algebraic equations which determine u_1 and u_2 as a function of q. The resultant equations are

$$\begin{aligned} &7.111\,\bar{u}_1^{\,3} + 8.191\,\bar{u}_1^{\,2}\bar{u}_2 + 35.79\,\bar{u}_1\bar{u}_2^{\,2} + 10.47\,\bar{u}_2^{\,3} \\ &-7.217\,\lambda^2\bar{u}_1\bar{u}_2 - 4.659\,\lambda^2\bar{u}_1^{\,2} - 9.349\,\lambda^2\bar{u}_2^{\,2} \\ &+ (.6984\,\lambda^4 + 36.38)\,\bar{u}_1 + (1.005\,\lambda^4 + 32.40)\,\bar{u}_2 \\ &\qquad - 1.189\,\lambda^4 q = 0 \end{aligned} \tag{5a}$$

$$\begin{aligned} &5.488\,\bar{u}_1^{\,3} - 13.09\,\bar{u}_1^{\,2}\bar{u}_2 + 17.11\,\bar{u}_1\bar{u}_2^{\,2} - 50.63\,\bar{u}_2^{\,3} \\ &+ 3.903\,\lambda^2\bar{u}_1\bar{u}_2 - 2.513\,\lambda^2\bar{u}_1^{\,2} + 5.006\,\lambda^2\bar{u}_2^{\,2} \\ &+ (.1010\,\lambda^4 + 17.11)\,\bar{u}_1 - (.03610\,\lambda^4 + 235.87)\,\bar{u}_2 = 0 \end{aligned} \tag{5b}$$

for $\mu = 1/3$.

Beginning at $\bar{u}_1 = \bar{u}_2 = 0$, equation 5b defines a continuous sequence of pairs u_1, u_2 which describe a series of neighboring equilibrium positions for the shell, which begins at the unloaded state. Equation 5a is used to calculate the value of q corresponding to a given pair. Tracing the u_1, u_2 curve from $u_1 = u_2 = q = 0$ we find q increases, reaches a maximum, and then decreases. This maximum value of q obviously defines the symmetrical buckling pressure, as there is no neighboring equilibrium position corresponding to $q = q_{max} + dq$. Following the shell beyond q_{max} we

find a minimum value of q and then the pressure again increases. This is the post buckling regime. The u_1, u_2, q curves for $\lambda = 5.5$ and $\lambda = 6$ are shown in Figs. 3 and 4. The $\lambda = 5.5$ curve is typical of the curves for $\lambda \leq 5.5$, while the $\lambda = 6$ curve is typical of the curves for $\lambda \geq 6$. The rapid transition between $\lambda = 5.5$ and $\lambda = 6$ is very striking, and in agreement with the sudden change of mode shape observed by Kaplan and Fung in this region.

The u_1, u_2 curves can be used in conjunction with Eq. 4 to plot the mode shapes as a function of q. This has been done for $\lambda = 5.5$ in Figure 5. The curves are in good agreement with the experimental measurements of Kaplan and Fung. The symmetrical buckling pressures (Fig. 1) agree very well with Budiansky for $\lambda \leq 5.5$, but diverge from his values for $\lambda \geq 6$. This is probably due to the increased importance of the third mode, $(1 + \cos \frac{3\pi}{b} r)$, for larger λ.

NON-SYMMETRICAL SOLUTION

The solution of the non-symmetrical equations is carried out in a similar manner. The function v(r) is assumed in the form

$$v(r) = v_1 r\,(r-b)^2 \tag{6}$$

This function satisfies the clamped condition $dv/dr = v = 0$ at $r = b$. Substituting this along with u(r) and G(r) from the symmetrical solution, in the right hand side of (3d) we can solve for H(r). The four constants of integration in H(r) are evaluated from the conditions of finite stress at $r = 0$ and zero inplane displacement, $z = y = 0$, at $r = b$. H(r), G(r), u(r), v(r) are now substituted into Eq. 3c and the equation is satisfied in the mean according to Galerkin's method. Since the equations are homogeneous, v_1 is a factor in every term of the resultant equation, so we obtain an equation of the form

$$v_1 \,[\, f(u_1,\, u_2) \,] = 0 \tag{7}$$

where $f(u_1, u_2)$ is a quadratic in u_1, u_2, whose coefficients depend on n. The desired condition for the existence of eigensolutions is the vanishing of the bracket.

The coefficients of $f(u_1;u_2)$ have been found for $n = 1$, and the resultant solution curve plotted in the u_1, u_2 plane for various λ. Intersections of this curve with the u_1, u_2 curve defined by the symmetrical solution represent bifurcation points in the symmetrical solution involving a branch with a non-symmetrical component approximated by $v(r) \cos\theta$. We find that no intersections occur between $q = 0$ and the symmetrical buckling pressure for $\lambda \leq 5.5$ (Fig. 3).

However, intersections do occur for $6 \leq \lambda \leq 8$ (Fig. 4) at a pressure of about 90% of our symmetrical buckling pressure. No conclusions can be drawn for $\lambda > 8$ because of limitations on the validity of the solutions for large λ.

Buckling pressure vs. λ according to our equations, and according to Budiansky, are shown in Fig. 1. Although our non-symmetrical, $n = 1$, buckling pressures fall below our symmetrical pressures, they are somewhat above the symmetrical buckling pressures of Budiansky. Improvements in both our non-symmetrical and symmetrical solutions will be required before any positive statements about the magnitude of the non-symmetrical buckling pressure can be made. However, the argument that non-symmetrical modes may participate in buckling at higher values of λ has been considerably strengthened by this work.

Our present efforts to improve our solutions are concentrated in three areas: a) Finding $f(u_1, u_2)$ for $n > 1$, b) finding non-symmetrical solutions for improved functions $v(r)$, c) extending the validity of our symmetrical solution to larger values of λ.

CONCLUSIONS

A two term Galerkin solution to the symmetrical, non-linear shallow spherical shell equations has been found which compares very well with experimental load deflection results and the buckling loads of Budiansky, for small values of λ. The results break down for larger λ because of the increased importance of higher modes in the deflection pattern. The effect on the buckling load of allowing non-symmetrical deflections has been investigated and found to be important at values of $\lambda > 5.5$. Work along these lines is being continued, with more positive results expected in the near future.

ACKNOWLEDGEMENT

This work was supported in part by the United States Air Force, through a grant from the Office of Scientific Research.

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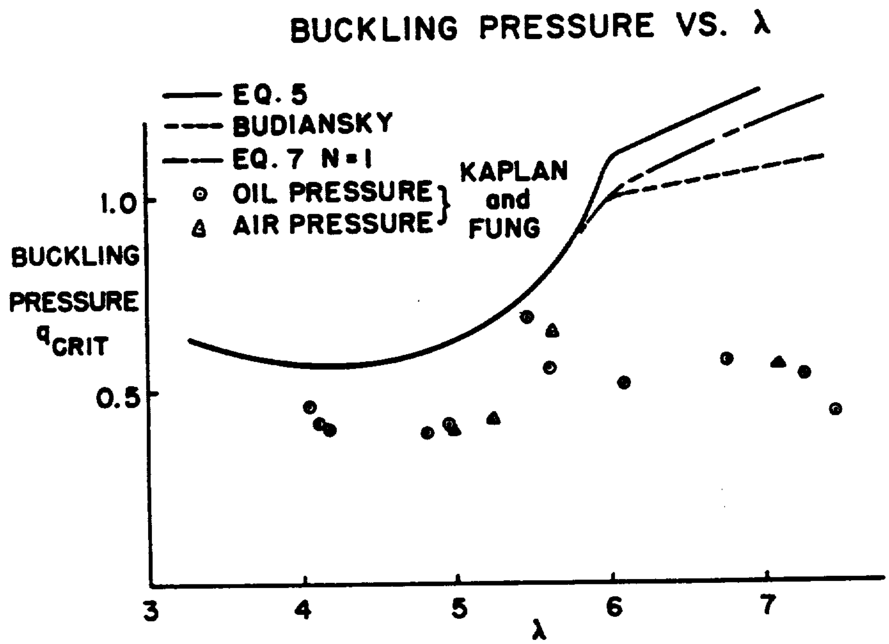

Figure 1

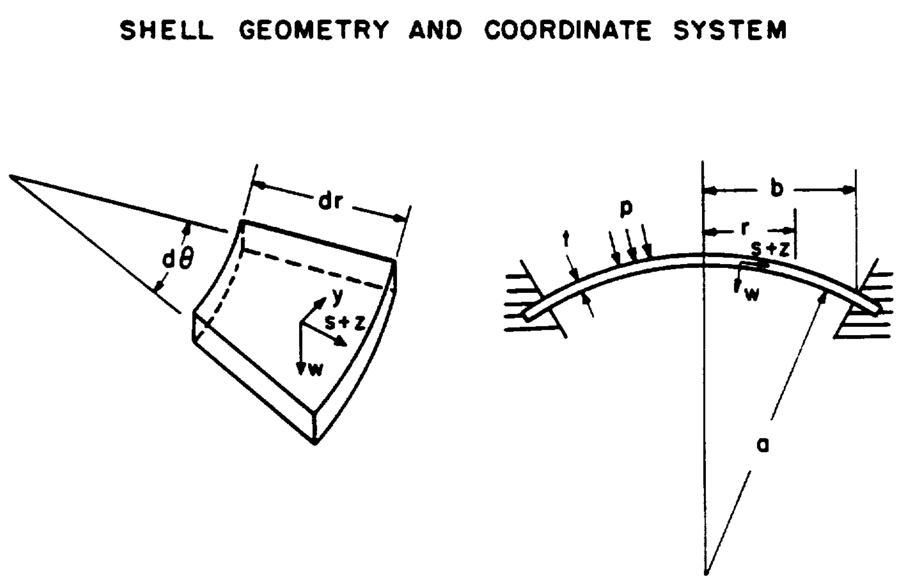

Figure 2

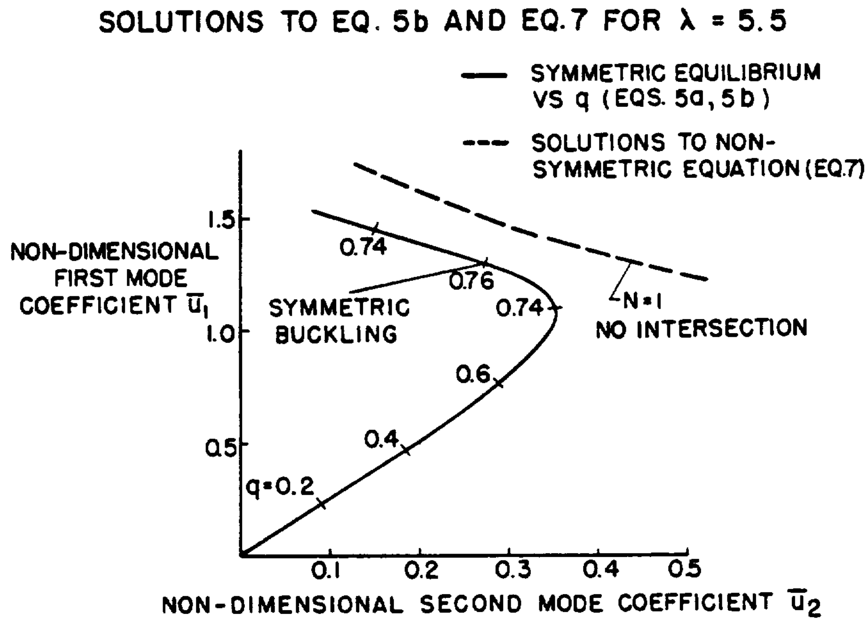

Figure 3

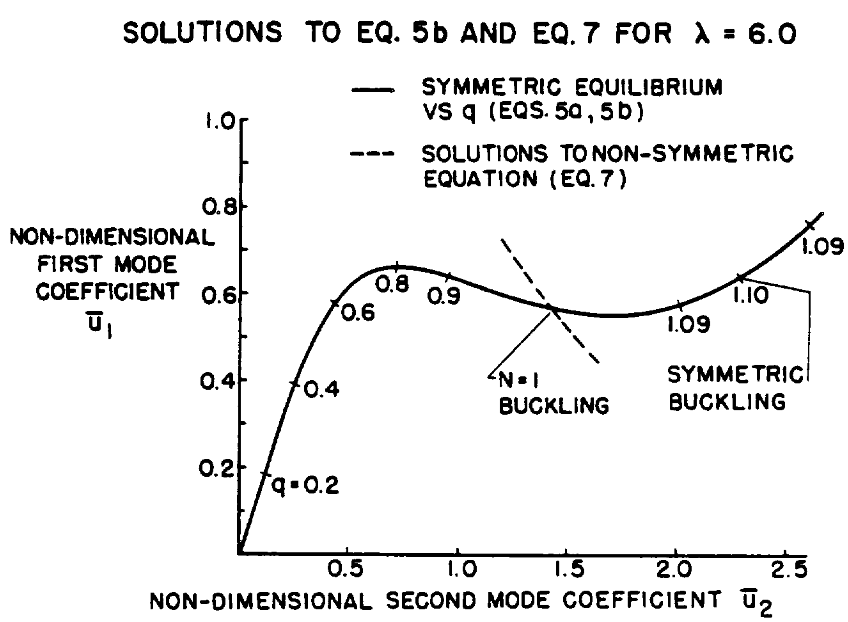

Figure 4

DEFLECTION VS. PRESSURE FOR λ = 5.5

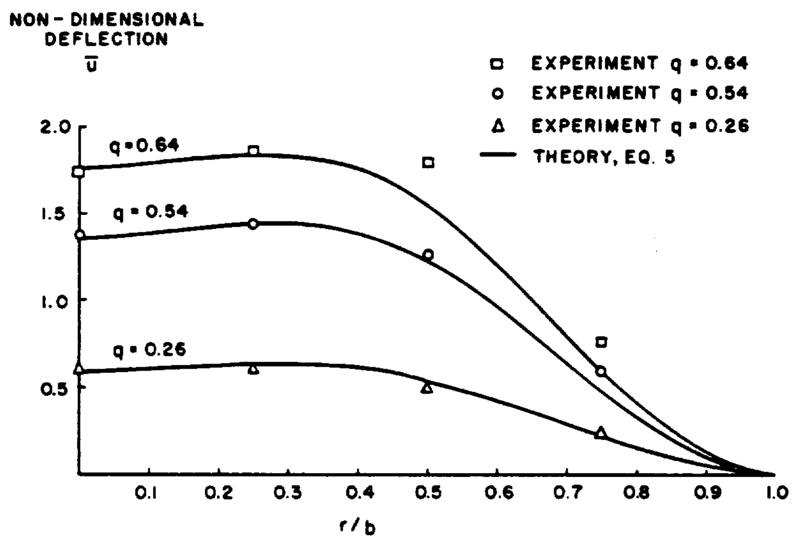

Figure 5